Laminar Flow and Heat Transfer in a Channel with Lateral Injection

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Nomenclature

h	= local heat-transfer coefficient, $= q_w/(T_w - T_o)$ or $q_w/(T_w - T_b)$
Nu	= Nusselt number based on channel height, $= hH/k$
P	= dimensionless local pressure, $= p/\rho v_w^2$
q_w	= local wall heat flux
Re	= injection Reynolds number, $= v_w H/\nu$
Re_c	= local channel Reynolds number, $= \bar{u}(x)H/\nu$
Re_o	= inlet channel Reynolds number, $= U_o H/\nu$
T_o	= injection fluid inlet temperature
T_w	= local heated wall temperature
U	= dimensionless streamwise velocity, $= u/\bar{u}$
U_o	= channel inlet velocity
u, v	= local velocities in the streamwise and lateral directions, respectively
$\bar{u}$	= local average velocity in the channel
V	= dimensionless cross-stream velocity, $= v/v_w$
v_w	= injection velocity
(X, Y)	= dimensionless streamwise and cross-stream coordinates, respectively, $= (x,y)/H$

Subscripts

b	= based on bulk temperature, T_b
fd	= fully developed regime
o	= based on fluid inlet temperature, T_o

Introduction

THE problem of flow and heat transfer in channels with porous walls finds applications in numerous industrial engineering situations. The traditional application has been the problem of blowing through a heated wall, which has proven effective in protecting materials from hot gases. The problem has been exhaustively treated both theoretically and experimentally.[1] However, the situation where a heated wall is cooled by uniform injection of cooler fluid through the opposite, porous wall has received only limited attention. Such a flow configuration finds particular relevance in the gas turbine industry. More recently, heat transfer in a channel with lateral injection has been proposed for use in advanced technology very large scale integrated (VLSI) circuits, where increased miniaturization is rendering conventional cooling techniques inadequate.

Flow and heat transfer in a channel with lateral injection have received some research attention. The general hydrodynamic nature of flow with injection has been treated theoretically using perturbation, series expansion, and similarity techniques.[2-5] A more recent study investigated the developing flow problem both numerically and experimentally.[6] Heat transfer and velocity profiles in the fully developed regime were obtained in the limit of large streamwise distance from the inlet. The objectives of this investigation were twofold: 1) quantify the development length in a heated channel with lateral injection of cooling fluid, and 2) demonstrate the existence of, and examine rigorously, the hydrodynamically and thermally fully developed regime.

Developing Flow Regime

The objective of the following brief analysis of the developing flow regime is to characterize the thermal entrance length in a heated channel with injection. Flow enters a channel of height H with uniform inlet velocity and temperature U_o and T_o, respectively (see inset, Fig. 1). Fluid is injected uniformly along the top boundary of the channel with velocity v_w and temperature T_o, whereas the lower wall is maintained at either isoflux (uniform q_w) or isothermal (uniform T_w) conditions. The classical parabolic differential equations governing the transport of mass, momentum, and energy are given elsewhere.[8] As defined, a positive injection velocity indicates that fluid is introduced into the channel. The conservation equations subject to the appropriate initial and boundary conditions were solved using the method of Patankar and Spalding,[7] a well-documented solution technique. Grid points were deployed in the cross-stream direction in a power law fashion using 300 nodes, clustering nodes near the channel walls. A constant axial step was used in the x direction ranging from $\Delta(x/H) = 10^{-3}$ to 10^{-6}, depending on Re and Re_o. The predicted local Nusselt number (based on local bulk temperature) normalized by its fully developed value, Nu/Nu_{fd}, is shown in Fig. 1 for several values of the injection Reynolds number. The predictions are for an isothermal wall thermal boundary condition and $Pr = 0.7$. The asymptotic behavior of the local heat transfer to a fully developed condition is clearly evident. For a given injection Reynolds number Re, the position at which the heat transfer is fully developed $(x/H)_{fd}$ decreases with decreasing inlet channel Reynolds number Re_o. This is consistent with previous analyses, which predict a more rapid hydrodynamic development in a channel with lateral fluid injection.[6,9] The reduced thermal development length in a channel with lateral injection underlines the importance of flow and heat transfer in the fully developed regime in engineering applications.

Fully Developed Regime

In the fully developed regime, the appropriately normalized velocity remains unchanged with axial position. Calculations for developing flow in the limit as $x \to \infty$ and experimental measurements of the streamwise velocity[6] indicate that either u_{max} or $\bar{u}$ may be used to normalize the velocity in the fully

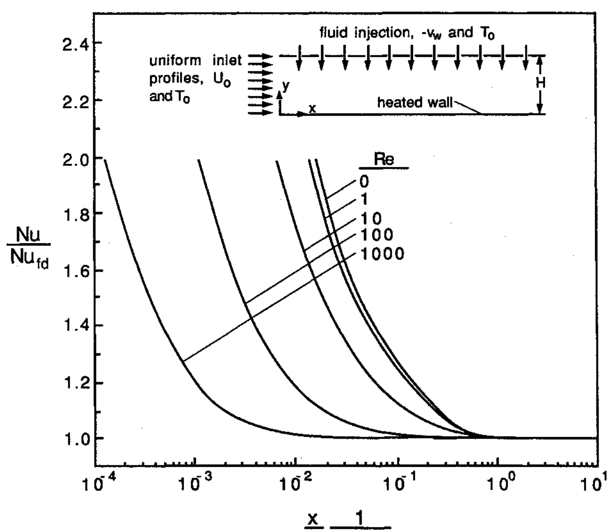

Fig. 1 Variation of local Nusselt number with position in the channel, x/HRe_oPr, for injection Reynolds numbers in the range $0 \le Re \le 10^3$.

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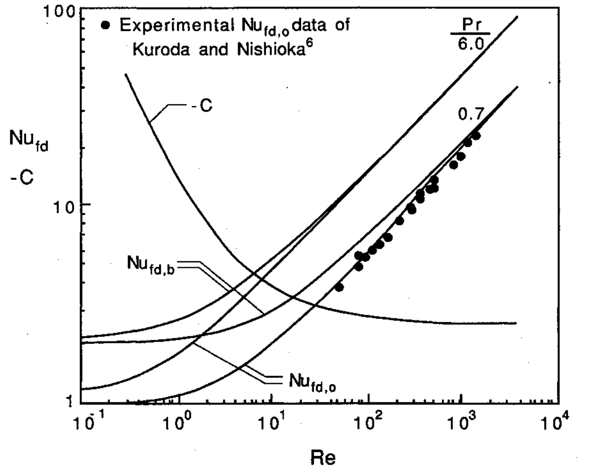

Fig. 2 Theoretical variation of fully developed Nusselt number Nu_{fd} and pressure gradient parameter C with injection Reynolds number Re.

developed regime because both increase linearly with streamwise location in the channel. Substituting $U = u/\bar{u}$ into the continuity equation, and recognizing that in the fully developed regime $\partial U/\partial x = 0$, yields

$$U\frac{\partial \bar{u}}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

Integrating the continuity equation over the channel cross section from $y = 0$ to $y = H$, evaluating the appropriate limits on u and v at the two walls and recognizing that the average channel velocity is a function only of x, $\bar{u} = \bar{u}(x)$, yields a global mass balance on the channel

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}x} = \frac{v_w}{H} \tag{2}$$

Substitution of Eq. (2) into Eq. (1) and nondimensionalizing using variables defined in the Nomenclature gives the continuity equation in the fully developed regime as

$$U = -\frac{\mathrm{d}V}{\mathrm{d}Y} \tag{3}$$

Since U is a function only of Y in the fully developed regime, Eq. (3) shows that $\mathrm{d}V/\mathrm{d}Y$ is independent of X. Further, the lateral velocity itself is also a function only of Y, $V = V(Y)$ in the fully developed regime because the no-slip and imposed v_w conditions at the two boundaries place natural limits on the magnitude of the lateral velocity component v. After substitution of $U = u/\bar{u}$ and nondimensionalization, the x momentum equation takes the form

$$U^2 + V\frac{\mathrm{d}U}{\mathrm{d}Y} = -\frac{Re}{Re_c}\frac{\mathrm{d}P}{\mathrm{d}X} + \frac{1}{Re}\frac{\mathrm{d}^2U}{\mathrm{d}Y^2} \tag{4}$$

where Re and Re_c are the injection and local channel Reynolds numbers, respectively. Since Re is constant, Re_c is a function only of X, and all other terms in Eq. (4) depend only on Y, one concludes that $-(Re/Re_c)\,\mathrm{d}P/\mathrm{d}X$ is invariant with X and Y, and is hereafter denoted C. Equation (4) can be expressed only in terms of V by substitution of Eq. (3),

$$(1/Re)V''' - VV'' + (V')^2 = C \tag{5}$$

where the primes denote differentiation with respect to Y. Equation (5) clearly demonstrates the existence of a hydrodynamically fully developed region since the vertical velocity component [and, therefore, according to Eq. (3), the appropriately normalized streamwise velocity component $U = u/\bar{u}$] is a function solely of vertical position in the channel, y/H. Equation (5) is virtually identical to the differential equation presented in a previous study.[5] Their analysis began with the substitution of a proposed stream function similarity variable of the form $\Psi = xv_w f(y/H)$ into both the horizontal and vertical equations of motion, which resulted in a single ordinary differential equation for the unknown function $f(y/H)$. Nothing was said about the developing or fully developed nature of the flow. By contrast, the approach used here has been the reduction of the parabolic differential equation governing streamwise momentum using the integral continuity equation and a normalized velocity profile that has been shown experimentally to be independent of x in the fully developed regime. The identical governing equations of the present analysis and Debruge and Han[5] suggest that their similarity solution is valid only in the fully developed regime.

To complete the specification of the problem, three boundary conditions on V are required. These boundary conditions may be stated as

$$Y = 0: \quad V = 0 \tag{6a}$$

$$V' = 0 \tag{6b}$$

$$Y = 1: \quad V = -1 \tag{6c}$$

Equations (6a) and (6c) come from the no-slip and specified injection velocity boundary conditions, respectively. Equation (6b) is arrived at by evaluating Eq. (3) at $Y = 0$. A fourth boundary condition on V, which will be used in the solution technique, may be stated by evaluating Eq. (3) also at $Y = 1$ as $V'(1) = 0$.

An established technique for transforming the boundary value problem into an initial value problem[11-13] was adopted for the solution of Eq. (5). A coordinate transformation is used of the form $V(Y) = A^bF(\eta)$ with $Y = A^a\eta$, where A is a parameter, and a and b are constants to be determined. Substituting the transformed variables into Eq. (5) and choosing the parameter and constants such that $A = Re$ and $C = A^{2(b-a)}$, the new differential equation to be solved becomes

$$F''' - FF' + (F')^2 = 1 \tag{7}$$

where the primes denote differentiation with respect to η. The boundary conditions in the transformed coordinates corresponding to Eq. (7) may be stated as

$$\eta = 0: \quad F = 0 \tag{8a}$$

$$F' = 0 \tag{8b}$$

$$\eta = A^{-a}: \quad F = -A^{-b} \tag{8c}$$

$$F' = 0 \tag{8d}$$

With this coordinte transformation, the solution procedure begins with specification of F and F' at $\eta = 0$ according to Eqs. (8a) and (8b). A value for F'' at $\eta = 0$ is also specified. A numerical integration is performed, marching through the solution domain in η until the value of F' returns to 0, specified by the boundary condition of Eq. (8d). At this point, the value of $\eta = \eta^* = A^{-a}$ and the corresponding value of $F(\eta^*) = F^* = -A^{-b}$. Hence, the values of the pressure gradient parameter and the injection Reynolds number are outcomes of the solution to the initial value problem subject to the specified value of F'' at $\eta = 0$. The value of Re and C may be determined from their definitions as stated in the foregoing as $C = A^{2(b-a)} = (\eta^*/F^*)^2$ and $Re = A^{-(b+a)} = -\eta^*F^*$. Additionally, the position and velocity components in the physical Y space may be determined from variables in the transformed space by the relations $Y = \eta/\eta^*$, $V = -F/F^*$, and

$U = (\eta^*/F^*)F'$. In practice, Eq. (7) was solved using a fourth-order Runge-Kutta numerical quadrature. Typical steps ranged from $\Delta\eta = 0.001$ to 0.0001, depending on the specified value of F''.

Consider now the energy transport of the lateral injection problem. If one integrates the differential energy equation over the channel cross section, the following arises:

$$\int_0^H \frac{\partial(uT)}{\partial x}\,dy + (vT)_{y=H} - (vT)_{y=0} = \frac{1}{\rho c_p} \times \left(k\frac{\partial T}{\partial y}\bigg|_{y=H} - k\frac{\partial T}{\partial y}\bigg|_{y=0}\right) \tag{9}$$

Under the assumption that advection is the only mode of energy transport associated with the fluid injection at $y = H$, the conduction flux there is identically zero, $\partial T/\partial y\,|_{y=H} = 0$. In addition, imposing the thermal boundary conditions at the heated wall, and after application of Leibnitz's formula, Eq. (9) reduces to

$$\frac{\partial}{\partial x}\int_0^H (uT)\,dy - v_w T_o = \frac{q_w}{\rho c_p} \tag{10}$$

Introducing the standard definition of the bulk temperature and recognizing that $\bar{u}$ and T_b are functions only of x gives the following equation for energy transport in the fully developed regime:

$$H\frac{d(\bar{u}T_b)}{dx} = \frac{q_w}{\rho c_p} + v_w T_o \tag{11}$$

For uniform heat flux at $y = 0$, the right side of Eq. (11) is constant since q_w is constant. The wall heat flux and, hence, right side of Eq. (11) are independent of x in the fully developed regime for given T_w as well. This is most easily observed first for impermeable walls and then for the heated channel including injection. For the impermeable wall case (with prescribed T_o at $y = H$), the fully developed Nusselt number has the same value,[8] $Nu_{fd,o} = 1.0$ ($Nu_{fd,b} = 2.0$), for both isoflux and isothermal wall heating at $y = 0$; the exact solution for the temperature profile in the channel for $v_w = 0$ is the same for both thermal boundary conditions and is given by $(T - T_w)/(T_o - T_w) = y/H$. If the wall at $y = H$ is made porous and cool fluid is injected, the fully developed Nusselt number must therefore asymptotically approach $Nu_{fd,o} = 1.0$ for vanishing injection Reynolds number regardless of specified thermal boundary condition at the heated wall. It then seems plausible that the temperature profiles in the channel (and corresponding Nusselt number) for a channel with mass injection would be identical for either imposed wall temperature or imposed heat flux at the heated wall. This is supported by the observation that for given T_w the boundary temperatures at both $y = 0$ and $y = H$ are fixed and the temperature gradient at the porous wall is constant at $\partial T/\partial y\,|_{y=H} = 0$. These fixed boundary temperatures and temperature gradient imply that the temperature profile and its gradient at $y = 0$ are independent of x. This was demonstrated earlier to be true for the case of no injection, and predictions have shown it to be true for $Re \neq 0$. The developing flow predictions previously discussed, and those for the fully developed regime to be presented, reveal that the fully developed Nusselt number is the same for both the prescribed wall temperature and wall heat flux thermal boundary conditions.

Since the entire right side of Eq. (11) is constant, one must conclude that $d(\bar{u}T_b)/dx$ is also constant. For uniform injection at $y = H$, Eq. (2) reveals that $\bar{u}$ is a linear function of x. It may then be stated that for channel flow with uniform lateral injection in the fully developed regime, $T_b \neq T_b(x)$, and is therefore constant.

The definition of the bulk temperature may be cast in a slightly different form by normalizing the streamwise velocity as defined, $U = u/\bar{u}$

$$T_b = \int_0^1 UT\,dY \tag{12}$$

Recognizing again the $U = U(Y)$ only, and that T_b = constant, one concludes from Eq. (12) that the temperature depends only on Y in the thermally fully developed regime, i.e., $\partial T/\partial x = 0$. This may be viewed as a degenerate condition of the traditional fully developed heat-transfer analysis for channel flow with impermeable walls,[14] $\partial[(T_w - T)/(T_w - T_b)]/\partial x = 0$.

Noting from the foregoing discussion that $\partial T/\partial x = 0$ in the thermally fully developed regime, the differential energy equation reduces to its dimensionless form

$$\frac{1}{RePr}T'' - VT' = 0 \tag{13}$$

Here, the primes denote differentiation with respect to Y. The appropriate boundary conditions for T are given as

$$Y = 0: \qquad T = T_w \text{ or } T' = -q_w H/k \tag{14a}$$

$$Y = 1: \qquad T = T_o \tag{14b}$$

Note that as has been stated previously, the solution for both isoflux and isothermal heated wall boundary conditions are identical in the fully developed regime. In practice, Eq. (7) was solved first to determine η^* and F^* for the specified F''. A second sweep through the η domain was then performed, calculating the value of the local vertical velocity component $V(\eta) = -F/F^*$. This local value of $V(\eta)$ was substituted into a concurrent Runge-Kutta solution of Eq. (13) using the same marching step $\Delta\eta$, yielding the corresponding local value of $T(\eta)$.

Predictions for temperature and axial velocity profiles across the channel were made for experimental conditions and results reported recently.[6] The experimental data corresponded to the hydrodynamically and thermally fully developed regime, which occurred at $x/H < 4.62$ for $Re_o = 0$ and all Re studied, $110 \leq Re \leq 1110$. The average discrepancy between the predicted cross-stream profiles of local dimensionless temperature and velocity, $(T - T_o)/(T_w - T_o)$ and U, respectively, was less than 8%.

Figure 2 shows the fully developed Nusselt number for $Pr = 0.7$ and 6.0, and the pressure gradient parameter C as a function of injection Reynolds number from the present analysis. Curves showing Nusselt number based both on fluid injection temperature, T_o and local bulk temperature T_b are shown. The analysis and experimental data of Kuroda and Nishioka[6] differ by an average of only 7%. As stated previously, the results for the constant heat flux and constant temperature thermal boundary conditions at $Y = 0$ are identical in the fully developed regime, which was verified by numerical computations for both. This is particularly important in applications where the exact nature of the thermal boundary condition at the heated wall is unknown. The vanishing injection asymptotes are $Nu_{fd,b} = 2.0$ and $Nu_{fd,o} = 1.0$, respectively, and are independent of fluid Prandtl number.

A least-squares regression of the analytical predictions reveals the relationship

$$Nu_{fd,o} = 0.72Re^{0.5}Pr^{0.38} \tag{15}$$

for $Re \geq 100$ and $0.7 \leq Pr \leq 100$. This relationship is very close to the analytical results of the similarity solution technique of Debruge and Han[5] and limiting-case numerical predictions of Kuroda and Nishioka.[6]

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